

Finite dimensional approximations for the symplectic vortex equations

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Abstract

In this paper we study Furuta's finite dimensional approximations for the flow lines of the Moment action functional for toric symplectic orbifolds. These are the analogon of the finite dimensional approximations for the Chern-Simons-Dirac functional studied by Kronheimer and Manolescu, see [11, 12]. The Moment action functional is the sum of Floer's action functional and a Lagrangian multiplier to a constraint given by the moment map. Its flow lines are the symplectic vortex equations on the cylinder.

We prove a compactness theorem specific for functions containing a Lagrange multiplier which allows us to define the Conley indices of the flow on the finite dimensional approximations. We show that these Conley are given by the Thom space of the normal bundle of toric map space introduced by Givental in [9]. Finally, we show how the moduli spaces of the Morse flow lines on the finite dimensional approximations are related to the moduli spaces of the symplectic vortex equations on the cylinder.

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1 Introduction

In [11, 12] finite dimensional approximation for the Seiberg Witten equation were studied. In this paper we will study their analogon for the symplectic vortex equations of toric symplectic orbifolds. The symplectic vortex equations were introduced in [1, 2, 13] and their solutions appeared in [5, 6] as flow lines in moment Floer homology. We prove that the finite dimensional approximations exist and show their relation to a finite dimensional model for the moduli spaces of Floer homology for toric symplectic orbifolds considered before by Givental in [9], see also [8, 10, 16]. To examine the relation between the original equations and their finite dimensional approximations we prove a cobordism theorem which relates the moduli spaces of the symplectic vortex equations to the moduli spaces of finite dimensional Morse theories.

The idea behind these finite dimensional approximations is the following. Assume that we have an action functional \mathcal{A} and a metric defined on an infinite dimensional space \mathcal{L} . Suppose further that there are finite dimensional spaces L_ν which approximate \mathcal{L} . We consider the sets $K_\nu \subset L_\nu$ which are given by the traces of all the flow lines of the action functional restricted to L_ν which converge at both ends, i.e.

$$K_\nu = \{y(\sigma) : \sigma \in \mathbb{R}, y \in C^\infty(\mathbb{R}, L_\nu), \partial_s y + \nabla \mathcal{A}|_{L_\nu} = 0, \exists \lim_{s \rightarrow \pm\infty} y(s)\}.$$

The sets K_ν are clearly invariant under the local flow of the restricted action functional. If they are also compact one can associate to them a Conley index I_ν . For good approximations one expects that the union of these Conley indices should contain all the information of the moduli space of all finite energy gradient trajectories of the action functional \mathcal{A} . As it was explained in [11] these Conley indices should be the building blocks of Furuta's pro-spectrum with parametrized universe which in the unobstructed case reduces to the Floer pro-spectrum introduced by Cohen, Jones and Segal in [3].

In [11, 12] Kronheimer and Manolescu studied finite dimensional approximations for the Chern-Simons-Dirac action functional which were used before by Furuta in his celebrated proof of the 10/8-conjecture, see [7]. In [12] Manolescu studied the case where the Chern-Simons-Dirac action functional was bounded on its critical set. Using this he could get compactness for the sets K_ν . In [11] Kronheimer and Manolescu considered the case where the Chern-Simons-Dirac action functional was not any more bounded on its critical set. In this case it is not clear if the sets K_ν are compact. However, Kronheimer and Manolescu could still define something provided a twist discovered by Furuta vanishes. In this paper we consider Furuta's finite dimensional approximations for the action functional of moment Floer homology. This action functional will in general not be bounded on its critical set. However, we will prove that we have compactness on the finite dimensional approximations.

Assume that a Lie group acts on a symplectic manifold by Hamiltonian isomorphisms, i.e. there exists a moment map μ for the action. Then the Moment action functional \mathcal{A} is defined as Floer's action functional together with

a Lagrangian multiplier to the constraint $\mu^{-1}(\tau)$, where τ is an element of the Lie algebra of the acting Lie group. In this paper we consider as symplectic manifold the complex vector space \mathbb{C}^n on which the torus T^k acts via a representation to the unitary group. We put the following standing assumption

(H) The moment map μ is proper and $\tau \in \mathfrak{t}^k$ is a regular value of μ .

Under this hypothesis the Marsden-Weinstein quotient $\mu^{-1}(\tau)/T^k$ is a compact toric symplectic orbifold.

We denote by \mathcal{L} the space of smooth loops on $\mathbb{C}^n \times \mathfrak{t}^k$ where \mathfrak{t}^k is the Lie algebra of the torus T^k . Then

$$\mathcal{A}: \mathcal{L} \rightarrow \mathbb{R}$$

and there is an action on \mathcal{L} of the gauge group \mathcal{H} consisting of smooth loops on the torus under which the differential of the action functional is invariant. There is a natural splitting

$$\mathcal{H} = H \times \mathcal{H}_0$$

such that the group $H \cong T^k \times \mathbb{Z}^k$ is finite, the group \mathcal{H}_0 is contractible and acts freely on \mathcal{L} , and the action functional is invariant under \mathcal{H}_0 . Since the gauge group is abelian we can find a global Coulomb section \mathcal{L}_0 in the principal \mathcal{H}_0 -bundle \mathcal{L} . This section is H -invariant, i.e. the following diagram commutes and is equivariant under the group H .

$$\begin{array}{ccc} & & \mathcal{L} \\ & \nearrow \iota & \downarrow \pi \\ \mathcal{L}_0 & \xrightarrow{c} & \mathcal{L}/\mathcal{H}_0 \end{array}$$

The L^2 -inner product endows \mathcal{L} with an \mathcal{H} -invariant metric g . We denote by \bar{g} its quotient metric on $\mathcal{L}/\mathcal{H}_0$. There are two natural H -invariant metrics on \mathcal{L}_0

$$g_0 = \iota^* g, \quad g_1 = c^* \bar{g}.$$

We will abbreviate

$$\mathcal{A}_0 = \iota^* \mathcal{A}.$$

Note that flow lines of the gradient of \mathcal{A} with respect to g are in one to one correspondence with flow lines of the gradient of \mathcal{A}_0 with respect to g_1 .

We approximate \mathcal{L}_0 by finite dimensional submanifolds L_ν . These are of the form

$$L_\nu \cong \mathbb{C}^{N_\nu} \times \mathfrak{t}^k.$$

The restriction of the action functional is linear in \mathfrak{t}^k , i.e. there exist smooth functions $f_\nu, h_\nu \in C^\infty(\mathbb{C}^{N_\nu}, \mathbb{R})$ such that

$$\mathcal{A}_0|_{L_\nu}(z, \eta) = f_\nu(z) + \langle h_\nu(z), \eta \rangle, \quad z \in \mathbb{C}^{N_\nu}, \quad \eta \in \mathfrak{t}^k.$$

In particular, the critical points of $\mathcal{A}_0|_{L_\nu}$ are critical points of f_ν to the Lagrangian constraint $h_\nu^{-1}(0)$. It turns out that the quotient of the zero sets of h_ν under the torus action are precisely the toric map spaces which were introduced before by Givental in [9] as a finite dimensional approximation to the moduli spaces of Floer homology for the symplectic orbifold $\mu^{-1}(\tau)/T^k$, see also [8, 10, 16].

We consider the finite energy gradient flow lines of $\mathcal{A}_0|_{L_\nu}$ with respect to the metrics g_0 and g_1 . We show that its traces are contained in a compact set which allows us to define the corresponding Conley indices $I_\nu^{g_0}$ and $I_\nu^{g_1}$. More precisely, our first main result is the following theorem.

Theorem A *The Conley indices $I_\nu^{g_0}$ and $I_\nu^{g_1}$ are well defined and isomorphic to the Thom space of the normal bundle of $h_\nu^{-1}(0)$ in \mathbb{C}^{N_ν} . In particular, they are isomorphic to each other.*

The idea of Theorem A is the following. For $r \in [0, 1]$ we consider the family of functions $F_{\nu,r} \in C^\infty(\mathbb{C}^{N_\nu} \times \mathfrak{t}^k, \mathbb{R})$ defined by

$$F_{\nu,r}(z, \eta) = rf_\nu(z) + \langle h_\nu(z), \eta \rangle.$$

Then $F_{\nu,1} = \mathcal{A}_0|_{L_\nu}$ and for $r = 0$ all flow lines of $F_{\nu,0}$ are constant and lie in $h_\nu^{-1}(0) \times \{0\}$.

We will denote by $\nabla_0 \mathcal{A}_0$ the gradient of \mathcal{A}_0 with respect to the metric g_0 . The submanifolds L_ν of \mathcal{L}_0 are totally geodesic with respect to the metric g_0 and if $p \in L_\nu$ then

$$\nabla_0 \mathcal{A}_0(p) \in T_p L_\nu.$$

Hence every flow line of the restriction of \mathcal{A}_0 to L_ν with respect to g_0 is actually a flow line of \mathcal{A}_0 on \mathcal{L} with respect to g_0 . We will show that each flow line of finite energy of $\nabla_0 \mathcal{A}_0$ is actually contained in some finite dimensional approximation.

Theorem B *Assume that $y: \mathbb{R} \rightarrow \mathcal{L}_0$ satisfies*

$$\partial_s y(s) + \nabla_0 \mathcal{A}_0(y(s)) = 0, \quad s \in \mathbb{R}, \quad \exists \lim_{s \rightarrow \pm\infty} y(s).$$

Then there exists L_ν such that the trace of y is contained entirely in L_ν , i.e.

$$y(s) \in L_\nu, \quad s \in \mathbb{R}.$$

Finally we prove a cobordism theorem between the moduli spaces of the gradient flow lines of \mathcal{A}_0 with respect to the metric g_0 and the ones with respect to the metric g_1 . More precisely, we show the following theorem.

Theorem C *The metrics g_0 and g_1 on \mathcal{L}_0 can be connected by a path of H -invariant metrics g_r for $r \in [0, 1]$ with the following property. Denote by $\nabla_r \mathcal{A}_0$*

the gradient of \mathcal{A}_0 with respect to the metric g_r . Assume that y_ν for $\nu \in \mathbb{N}$ is a sequence of solutions of

$$\partial_s y_\nu(s) + \nabla_{r_\nu} \mathcal{A}_0(y(s)) = 0, \quad s \in \mathbb{R}, \quad r_\nu \in [0, 1]$$

whose energy is uniformly bounded. Then there exists a subsequence ν_j , a sequence of gauge transformations $h_j \in H$ and a flow line y of $\nabla_r \mathcal{A}_0$ for $r \in [0, 1]$, such that $(h_j)_* y_{\nu_j}$ converges with respect to the C_{loc}^∞ -topology to y , i.e.

$$(h_j)_* y_{\nu_j} \xrightarrow{j \rightarrow \infty} y.$$

The metric g_r for $r \in (0, 1]$ will be constructed in the following way. We deform the action of \mathcal{H}_0 on \mathcal{L} in an H -invariant way by free actions for which \mathcal{L}_0 is still a global section. The metric g_r is then defined by pulling back the quotient metric on $\mathcal{L}/_r \mathcal{H}_0$ induced by the original L^2 -metric g on \mathcal{L} .

This paper is organized as follows. In Section 2 we recall the definition of toric symplectic orbifolds and prove a regularity criterion which will allow us later to prove the regularity of Givental's toric map spaces.

In Section 3 we recall the definition of the Moment action functional. We define the global Coulomb section on the loop space and describe the two metrics g_0 and g_1 . We compute the gradient of the action functional with respect to both metrics and prove an estimate for the two metrics which will allow us later to conclude that the finite dimensional approximations are geodesically complete.

In Section 4 we introduce the finite dimensional approximations. We prove a regularity result for Givental's toric map space and show that the finite dimensional approximations are geodesically complete for both metrics. These properties will be crucial to prove that the Conley index is well defined.

In Section 5 we prove Theorem A. We first introduce the Conley indices. Then we will prove a more general theorem which allows us to compute the Conley index for a large class of functions which contain some Lagrange multiplier part. Theorem A will follow as an easy Corollary of this more general theorem and the results obtained in Section 4.

In Section 6 we will prove Theorems B and C.

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2 Toric symplectic orbifolds

Assume that for $k \leq n$ the torus $T^k = \{e^{iv} : v \in \mathbb{R}^k\}$ acts on the complex vector space \mathbb{C}^n via the action

$$\rho(e^{iv})z = e^{iAv}z, \quad z \in \mathbb{C}^n, \quad v \in \mathbb{R}^k$$

for some $(n \times k)$ -matrix A with integer entries. We endow the Lie algebra of the torus

$$\text{Lie}(T^k) = \mathfrak{t}^k = i\mathbb{R}^k$$

with its standard inner product. The action of the torus on \mathbb{C}^n is Hamiltonian with respect to the standard symplectic structure $\omega = \sum_{i=1}^n dx_i \wedge dy_i$. Denoting by A^T the transposed matrix of A a moment map $\mu: \mathbb{C}^n \rightarrow \mathfrak{t}^k$ is given by

$$\mu(z) = \frac{i}{2}A^T w, \quad w = \begin{pmatrix} |z_1|^2 \\ \vdots \\ |z_n|^2 \end{pmatrix}, \quad (1)$$

i.e.

$$d\langle \mu, \xi \rangle = \iota_{X_\xi} \omega, \quad \xi \in \mathfrak{t}^k$$

for the vector field X_ξ on \mathbb{C}^n given by the infinitesimal action

$$X_\xi(z) = \dot{\rho}(\xi)(z), \quad z \in \mathbb{C}^n.$$

We will assume throughout this paper hypothesis (H), i.e. the moment map is proper and $\tau \in \mathfrak{t}^k$ is a regular value of the moment map. It follows from (H) that the Marsden-Weinstein quotient

$$\mathbb{C}^n / T^k = \mu^{-1}(\tau) / T^k$$

is a compact symplectic orbifold of dimension

$$\dim(\mathbb{C}^n / T^k) = 2(n - k),$$

where the symplectic structure is induced from the standard symplectic structure on \mathbb{C}^n .

Remark 2.1 *By (3) the moment map is proper if and only if each column of the matrix A has fixed sign, i.e. for each fixed $j \in \{1, \dots, k\}$ the sign of $A_{\ell j}$ is independent of $\ell \in \{1, \dots, n\}$.*

We next examine which values $\tau \in \mathfrak{t}^k$ are regular. In order to do that we first introduce some notation. Let $\mathcal{J} = \mathcal{J}_n$ be the set of all strictly monotone functions $\phi: \{1, \dots, n_1\} \rightarrow \{1, \dots, n\}$ for a positive integer $n_1 \leq n$. For each $\phi \in \mathcal{J}$ we define the $n_1 \times k$ -matrix A_ϕ by

$$(A_\phi)_{j\ell} := A_{j\phi(\ell)}, \quad 1 \leq \ell \leq n_1, \quad 1 \leq j \leq k.$$

We set

$$\mathcal{A} := \{A_\phi : \phi \in \mathcal{J}, \text{rk}(A_\phi) < k\}. \quad (2)$$

Lemma 2.2 *The element of the Lie algebra $\tau \in \mathfrak{t}^k$ is an irregular value of the moment map μ , iff $\tau = 0$ or $\tau = A_\phi w$ where $A_\phi \in \mathcal{A}$ and the real vector w has the property that each entry is nonnegative, i.e. $w_j \geq 0$ for $1 \leq j \leq n_1$.*

Proof: Assume that $\tau = A_\phi w$ where A_ϕ and w satisfy the assumptions of the lemma. Choose a complex n -vector z such that

$$|z_{\phi(\ell)}|^2 = 2w_\ell, \quad 1 \leq \ell \leq n_1, \quad z_m = 0, \quad m \in \{1, \dots, n\} \setminus \text{im}(\phi).$$

Then

$$\mu(z) = \tau$$

and by (3)

$$\text{rk}(d\mu(z)) \leq \text{rk}(A_\phi) < k$$

which shows that τ is a nonregular value of the moment map.

Now assume that $\tau \neq 0$ is a nonregular value of the moment map and choose $z \in \mathbb{C}^n$ such that $\mu(z) = \tau$ and $\text{rk}(d\mu(z)) < k$. Define $\phi \in \mathcal{I}$ by the property that

$$\phi(1) := \min\{j \in \{1, \dots, n\} : z_j \neq 0\},$$

$$\phi(\ell) := \min\{j \in \{\phi(\ell-1)+1, \dots, n\} : z_j \neq 0\}, \quad 2 \leq \ell \leq \#\{j \in \{1, \dots, n\} : z_j \neq 0\}.$$

Define the nonnegative vector w by

$$w_\ell := \frac{1}{2}|z_{\phi(\ell)}|^2, \quad 1 \leq \ell \leq \#\{j \in \{1, \dots, n\} : z_j \neq 0\}.$$

It follows that $\tau = A_\phi w$ which proves the lemma. \square

The following Corollary of Lemma 2.2 will be used later on to prove regularity of Givental's spaces.

Corollary 2.3 *Assume that the torus T^k acts on \mathbb{C}^n and \mathbb{C}^m with $n \times k$ -matrix A respectively $m \times k$ -matrix B such that for each column vector A_ℓ with $\ell \in \{1, \dots, n\}$ of the matrix A there exists a column vector $B_{\ell'}$ of the matrix B and a positive number λ_ℓ such that*

$$A_\ell = \lambda_\ell B_{\ell'}.$$

Suppose that $\tau \in \mathfrak{t}^k$ is an irregular value of the moment map μ_A associated to the matrix A . Then τ is also an irregular value of the moment map μ_B associated to the matrix B .

Proof: We may assume that $\tau \neq 0$. In this case τ is an irregular nonzero value of the moment map μ_A so that there exists by Lemma 2.2 an element $\phi \in \mathcal{I}_n$, i.e. a strictly monotone function from $\{1, \dots, n_1\} \rightarrow \{1, \dots, n\}$ for $n_1 \leq n$, and a vector w with nonnegative entries such $\text{rk}(A_\phi) < k$ and $A_\phi w = \tau$. Choose functions $\lambda: \{1, \dots, n\} \rightarrow \mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\}$ and $\rho: \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ such that

$$A_\ell = \lambda(\ell) B_{\rho(\ell)} \tag{3}$$

which exist by the assumption of the Corollary. Let $m_1 \leq n_1$ be the cardinality of the set $\rho \circ \phi\{1, \dots, n_1\}$ and define the strictly monotone function $\tilde{\phi} \in \mathcal{I}_m$ from $\{1, \dots, m_1\}$ to $\{1, \dots, m\}$ recursively by

$$\tilde{\phi}(\ell) = \min(\rho \circ \phi\{1, \dots, n_1\} \setminus \tilde{\phi}\{1, \dots, \ell - 1\}), \quad \ell \in \{1, \dots, m_1\}.$$

It follows from (3) that

$$\text{rk}(B_{\tilde{\phi}}) = \text{rk}(A_{\phi}) < k.$$

Define for $1 \leq j \leq m_1$

$$\tilde{w}_j := \sum_{k \in \rho^{-1}\tilde{\phi}(j)} \lambda(k) w_{\phi^{-1}(k)}.$$

Then \tilde{w}_j is nonnegative and we calculate

$$\begin{aligned} B_{\tilde{\phi}} \tilde{w} &= \sum_{j=1}^{m_1} B_{\tilde{\phi}(j)} \tilde{w}_j \\ &= \sum_{j=1}^{m_1} B_{\tilde{\phi}(j)} \left(\sum_{k \in \rho^{-1}\tilde{\phi}(j)} \lambda(k) w_{\phi^{-1}(k)} \right) \\ &= \sum_{j=1}^{m_1} \left(\sum_{k \in \rho^{-1}\tilde{\phi}(j)} \lambda(k) w_{\phi^{-1}(k)} B_{\rho(k)} \right) \\ &= \sum_{j=1}^{m_1} \left(\sum_{k \in \rho^{-1}\tilde{\phi}(j)} A_k w_{\phi^{-1}(k)} \right) \\ &= \sum_{i=1}^{n_1} A_{\phi(i)} w_i \\ &= A_{\phi} w \\ &= \tau. \end{aligned}$$

Now Lemma 2.2 implies that τ is an irregular value of μ_B . This proves the Corollary. \square

3 Moment Floer homology

Let \mathcal{L} be the loop space

$$\mathcal{L} := C^\infty(S^1, \mathbb{C}^n \times \mathfrak{t}^k),$$

where

$$S^1 = \mathbb{R}/\mathbb{Z}$$

is the circle. The gauge group

$$\mathcal{H} = C^\infty(S^1, T^k)$$

acts on \mathcal{L} by

$$h_*(z, \eta) = (\rho(h)z, \eta - h^{-1}\partial_t h), \quad h \in \mathcal{H}, (z, \eta) \in \mathcal{L}.$$

The action functional for moment Floer homology $\mathcal{A}: \mathcal{L} \rightarrow \mathbb{R}$ is given by

$$\mathcal{A}(z, \eta) := \int_0^1 \lambda(z)(\partial_t z) + \int_0^1 \langle \mu(z(t)) - \tau, \eta(t) \rangle dt.$$

Here λ denotes the Liouville 1-form

$$\lambda = \sum_{i=1}^n y_i dx_i, \quad d\lambda = -\omega.$$

The first integral on the right hand side is Floer's action functional on the space of loops in \mathbb{C}^n . One may think of η in the second integral as Lagrange multiplier to the constraint $\mu(z) = \tau$.

Borrowing notation from [12] we introduce the subgroup $\mathcal{H}_0 \subset \mathcal{H}$ of "normalized gauge transformations" consisting of $h \in \mathcal{H}$ for which there exists $\xi \in C^\infty(S^1, \mathfrak{t}^k)$ with $\int_0^1 \xi dt = 0$ such that

$$h(t) = \exp(\xi(t)), \quad t \in S^1.$$

The gauge group \mathcal{H} can be written as

$$\mathcal{H} = \mathcal{H}_0 \times H$$

where the finite dimensional group

$$H = \{h \in \mathcal{H} : \partial_t(h^{-1}\partial_t h) = 0\}$$

consists of elements $h = h_0 \cdot e^{2\pi i v t}$ with $h_0 \in T^k$ and $v \in \mathbb{Z}^k$. There is a natural isomorphism

$$H \cong T^k \times \mathbb{Z}^k$$

given by

$$h \mapsto \left(h(0), \frac{1}{2\pi i} (h^{-1}\partial_t h)(0) \right), \quad h \in H.$$

The action functional is invariant under \mathcal{H}_0 and \mathcal{H}_0 acts freely on \mathcal{L} . Moreover, using a gauge transformation $h \in \mathcal{H}_0$ we can assume that $h_*\eta$ is constant. This gauge fixing is the analogon of the Coulomb gauge fixing considered in [11, 12]. In particular, the Coulomb gauge gives a natural bijection

$$\mathcal{L}/\mathcal{H}_0 \cong C^\infty(S^1, \mathbb{C}^n) \times \mathfrak{t}^k =: \mathcal{L}_0.$$

We consider as in [11, 12] the restriction of the action functional \mathcal{A} to the space \mathcal{L}_0

$$\mathcal{A}_0(z, \eta) = \mathcal{A}|_{\mathcal{L}_0}(z, \eta) = \int_0^1 \lambda(z)(\partial_t z) + \left\langle \eta, \int_0^1 \mu(z(t)) dt - \tau \right\rangle.$$

Note that for \mathcal{A}_0 the Lagrangian multiplier η gives only a constraint on the mean value of $\mu(z)$.

On \mathcal{L} we have a natural \mathcal{H}_0 -invariant L^2 -metric g defined by integration

$$g((\hat{z}_1, \hat{\eta}_1), (\hat{z}_2, \hat{\eta}_2)) = \int_0^1 \langle \hat{z}_1, \hat{z}_2 \rangle dt + \int_0^1 \langle \hat{\eta}_1, \hat{\eta}_2 \rangle dt$$

where

$$(\hat{z}_1, \hat{\eta}_1), (\hat{z}_2, \hat{\eta}_2) \in T_{(z, \eta)} \mathcal{L} \cong \mathcal{L}, \quad (z, \eta) \in \mathcal{L}.$$

Using the L^2 -metric on \mathcal{L} we can define two different metrics on \mathcal{L}_0 . The first one g_0 is given by the pullback of the inclusions $\iota: \mathcal{L}_0 \rightarrow \mathcal{L}$, i.e. $g_0 := \iota^* g$. The second one g_1 is the induced metric on the quotient $\mathcal{L}_0 \cong \mathcal{L} / \mathcal{H}_0$. If $(z, \eta) \in \mathcal{L}_0$ and $(\hat{z}_1, \hat{\eta}_1), (\hat{z}_2, \hat{\eta}_2) \in T_{(z, \eta)} \mathcal{L}_0 \cong \mathcal{L}_0$ then the first metric is again given by an inner product on the vector space \mathcal{L}_0

$$g_0((\hat{z}_1, \hat{\eta}_1), (\hat{z}_2, \hat{\eta}_2)) = \int_0^1 \langle \hat{z}_1, \hat{z}_2 \rangle dt + \langle \hat{\eta}_1, \hat{\eta}_2 \rangle.$$

To give a formula for the second metric we first introduce some notation. We define for $z \in \mathbb{C}^n$ the linear map $L_z: \mathfrak{t}^k \rightarrow T_z \mathbb{C}^n \cong \mathbb{C}^n$ by

$$L_z \xi := X_\xi(z) = \dot{\rho}(\xi)z, \quad \xi \in \mathfrak{t}^k$$

and denote by $L_z^*: T_z \mathbb{C}^n \rightarrow \mathfrak{t}^k$ its adjoint with respect to the standard inner products on the Lie algebra $\mathfrak{t}^k = i\mathbb{R}^n$ and \mathbb{C}^n . If $\xi \in \text{Lie}(\mathcal{H}_0)$, i.e. $\xi \in C^\infty(S^1, \mathfrak{t}^k)$ and $\int_0^1 \xi dt = 0$, then the infinitesimal action of ξ on $(z, \eta) \in \mathcal{L}$ is given by

$$L_{(z, \eta)} \xi = (L_z \xi, -\partial_t \xi). \quad (4)$$

In particular, $(\hat{z}, \hat{\eta}) \in T_z \mathcal{L} \cong \mathcal{L}$ lies orthogonal to the infinitesimal gauge action iff

$$\frac{d}{dt} \left(L_z^* \hat{z} + \partial_t \hat{\eta} \right) = 0.$$

The formula for the second metric on \mathcal{L}_0 can now be written as

$$\begin{aligned} g_1((\hat{z}_1, \hat{\eta}_1), (\hat{z}_2, \hat{\eta}_2)) &= g((\hat{z}_1 - L_z \xi_1, \hat{\eta}_1 + \partial_t \xi_1), (\hat{z}_2 - L_z \xi_2, \hat{\eta}_2 + \partial_t \xi_2)) \quad (5) \\ &= \int_0^1 (\langle \hat{z}_1 - L_z \xi_1, \hat{z}_2 - L_z \xi_2 \rangle + \langle \partial_t \xi_1, \partial_t \xi_2 \rangle) dt \\ &\quad + \langle \hat{\eta}_1, \hat{\eta}_2 \rangle \end{aligned}$$

where the Lie algebra elements $\xi_1, \xi_2 \in \text{Lie}(\mathcal{H}_0)$ are determined by

$$\frac{d}{dt} \left(L_z^* \hat{z}_i - L_z^* L_z \xi_i + \partial_t^2 \xi_i \right) = 0, \quad \int_0^1 \xi_i dt = 0, \quad i \in \{1, 2\}.$$

We denote by $\|\cdot\|_i$ the length of a tangent vector with respect to the metric g_i for $i \in \{0, 1\}$. We clearly have

$$\|\cdot\|_1 \leq \|\cdot\|_0.$$

In the following proposition we will estimate the metric g_0 by the metric g_1 . We will use this estimate later to prove that the finite dimensional approximations of \mathcal{L}_0 are totally geodesic with respect to g_1 .

Proposition 3.1 *There exists a constant $c > 0$ such that*

$$\|\cdot\|_0 \leq c(1 + \|z\|_{L_2}) \cdot \|\cdot\|_1.$$

Proof: Let $(z, \eta) \in \mathcal{L}_0$, $(\hat{z}, \hat{\eta}) \in T_{(z, \eta)}\mathcal{L}_0 \cong \mathcal{L}_0$ and define $\xi \in \text{Lie}(\mathcal{H}_0)$ by

$$\frac{d}{dt} \left(L_z^* \hat{z} - L_z^* L_z \xi + \partial_t^2 \xi \right) = 0, \quad \int_0^1 \xi dt = 0. \quad (6)$$

We calculate

$$\begin{aligned} 0 &= \int_0^1 \langle L_z^* \hat{z} - L_z^* L_z \xi + \partial_t^2 \xi, \xi \rangle dt \\ &= \int_0^1 \langle \hat{z}, L_z \xi \rangle dt - \int_0^1 \langle L_z \xi, L_z \xi \rangle dt - \int_0^1 \langle \partial_t \xi, \partial_t \xi \rangle dt \end{aligned}$$

from which we obtain by the Cauchy-Schwarz inequality

$$\|L_z \xi\|_{L_2}^2 + \|\partial_t \xi\|_{L_2}^2 \leq \|\hat{z}\|_{L_2} \|L_z \xi\|_{L_2}.$$

This implies

$$\|L_z \xi\|_{L_2} \leq \frac{\|L_z \xi\|_{L_2}^2}{\|L_z \xi\|_{L_2}^2 + \|\partial_t \xi\|_{L_2}^2} \|\hat{z}\|_{L_2}. \quad (7)$$

Using the second equation in (6), partial integration and Hölders inequality we obtain as a special case of Poincaré's inequality

$$\|\xi\|_{L_2} \leq \|\partial_t \xi\|_{L_2}. \quad (8)$$

The formula $L_z \xi = \dot{\rho}(\xi)z$ together with Hölders inequality implies that there exists a constant $c_0 > 0$ such that

$$\|L_z \xi\|_{L_2} \leq c_0 \|z\|_{L_2} \|\xi\|_{L_2}. \quad (9)$$

Using inequalities (7), (8), and (9) we obtain the inequality

$$\|L_z \xi\|_{L_2} \leq \frac{c_0^2 \|z\|_{L_2}^2}{c_0^2 \|z\|_{L_2}^2 + 1} \|\hat{z}\|_{L_2}. \quad (10)$$

We now estimate using (5) and (10)

$$\begin{aligned} \|(\hat{z}, \hat{\eta})\|_1^2 &\geq \|\hat{z} - L_z \xi\|_{L_2}^2 + \|\hat{\eta}\|_{L_2}^2 \\ &\geq \|\hat{z}\|_{L_2}^2 - \|L_z \xi\|_{L_2}^2 + \|\hat{\eta}\|_{L_2}^2 \\ &\geq \|\hat{z}\|_{L_2}^2 - \left(\frac{c_0^2 \|z\|_{L_2}}{c_0^2 \|z\|_{L_2}^2 + 1} \right)^2 \|\hat{z}\|_{L_2}^2 + \|\hat{\eta}\|_{L_2}^2 \\ &\geq \|\hat{z}\|_{L_2}^2 - \frac{c_0^2 \|z\|_{L_2}}{c_0^2 \|z\|_{L_2}^2 + 1} \|\hat{z}\|_{L_2}^2 + \|\hat{\eta}\|_{L_2}^2 \\ &\geq \frac{1}{c_0^2 \|z\|_{L_2}^2 + 1} \|(\hat{z}, \hat{\eta})\|_0. \end{aligned}$$

The Proposition follows now with $c = \max\{c_0, 1\}$. \square

We denote by $\nabla_0 \mathcal{A}_0$ the gradient of \mathcal{A}_0 on \mathcal{L}_0 with respect to the metric g_0 and by $\nabla_1 \mathcal{A}_0$ the gradient with respect to the metric g_1 . We will use the notation

$$\bar{\mu}(z) := \int_0^1 \mu(z(t)) dt.$$

Proposition 3.2 *The two gradients are given by*

$$\nabla_0 \mathcal{A}_0(z, \eta) = (i\partial_t z + iL_z \eta, \bar{\mu}(z) - \tau) \quad (11)$$

and

$$\nabla_1 \mathcal{A}_0(z, \eta) = (i\partial_t z + iL_z \eta + L_z \xi, \bar{\mu}(z) - \tau) \quad (12)$$

where $\xi \in \text{Lie}(\mathcal{H}_0)$ is determined by

$$\partial_t \xi(t) = \mu(z(t)) - \bar{\mu}(z), \quad \int_0^1 \xi dt = 0. \quad (13)$$

Proof: To determine the first gradient we compute for $(\hat{z}, \hat{\eta}) \in T_{(z, \eta)} \mathcal{L}_0 \cong \mathcal{L}_0$

$$\begin{aligned} d\mathcal{A}_0(z, \eta)(\hat{z}, \hat{\eta}) &= \int_0^1 d\lambda(z)(\hat{z}, \partial_t z) + \langle \eta, \int_0^1 d\mu(z(t)) \hat{z}(t) dt \rangle + \langle \hat{\eta}, \bar{\mu}(z) - \tau \rangle \\ &= - \int_0^1 \omega(\hat{z}, \partial_t z) + \int_0^1 \omega(X_\eta(z), \hat{z}) + \langle \hat{\eta}, \bar{\mu}(z) - \tau \rangle \\ &= \langle (i\partial_t z + iL_z \eta, \bar{\mu}(z) - \tau), (\hat{z}, \hat{\eta}) \rangle_1 \end{aligned}$$

which implies (11).

To compute the second gradient we make use of the fact that the action functional \mathcal{A} on \mathcal{L} is invariant under \mathcal{H}_0 and hence its gradient $\nabla \mathcal{A}$ is orthogonal to the infinitesimal gauge action. For $\zeta = (\hat{z}, \hat{\eta}) \in T_{(z, \eta)} \mathcal{L}_0$ we denote by $\xi_\zeta \in \text{Lie}(\mathcal{H}_0)$ the Lie algebra element which is determined by

$$\frac{d}{dt} \left(L_z^* \hat{z} - L_z^* L_z \xi_\zeta + \partial_t^2 \xi_\zeta \right) = 0, \quad \int_0^1 \xi_\zeta dt = 0.$$

We will abbreviate

$$\xi = \xi_{\nabla_1 \mathcal{A}_0(z, \eta)}.$$

We compute for the metric $\langle \cdot, \cdot \rangle$ on \mathcal{L}

$$\begin{aligned} \langle \nabla_1 \mathcal{A}_0(z, \eta) - L_{(z, \eta)} \xi, \zeta - L_{(z, \eta)} \xi_\zeta \rangle &= \langle \nabla_2 \mathcal{A}_0(z, \eta), \zeta \rangle_2 \\ &= d\mathcal{A}_0(z, \eta) \zeta \\ &= d\mathcal{A}(z, \eta) \zeta \\ &= \langle \nabla \mathcal{A}(z, \eta), \zeta \rangle \\ &= \langle \nabla \mathcal{A}(z, \eta), \zeta - L_{(z, \eta)} \xi_\zeta \rangle \end{aligned}$$

which implies

$$\nabla_1 \mathcal{A}_0(z, \eta) - L_{(z, \eta)} \xi = \nabla \mathcal{A}(z, \eta). \quad (14)$$

By a computation similar to the computation of $\nabla_0 \mathcal{A}_0$ one gets

$$\nabla \mathcal{A}(z, \eta) = (i\partial_t z + iL_z \eta, \mu(z) - \tau). \quad (15)$$

Using (4), (14), and (15) we conclude that $\mu(z) - \tau - \partial_t \xi$ is independent of the t -variable. Observing that $\xi(0) = \xi(1)$ we obtain the formula (13) for ξ . Using again (14) and (15) we are able to deduce the formula (12) for the second gradient. This completes the proof of the proposition. \square

4 Finite dimensional approximation

Formulas (11) and (12) show that the gradients of Floer's action functional are a zero'th order perturbation of the Cauchy-Riemann operator. Following [7, 11, 12] we will approximate the loop space by the eigenspaces of the first order part of the gradient, which leads in our case to Fourier approximation.

For $1 \leq j \leq n$ let $m_j^- \leq m_j^+$ be integers. We consider the finite dimensional complex vector space

$$V = V_{\{m_j^-, m_j^+\}_{1 \leq j \leq n}}$$

consisting of finite Fourier series $z = (z_1, \dots, z_n) \in C^\infty(S^1, \mathbb{C}^n)$ for which there exist complex numbers z_{jm} where $1 \leq j \leq n$ and $m_j^- \leq m \leq m_j^+$ such that

$$z_j(t) = \sum_{m=m_j^-}^{m_j^+} z_{jm} e^{2\pi i m t}, \quad j \in \{1, \dots, n\}, \quad t \in S^1.$$

The action of the torus T^k on \mathbb{C}^n induces an action of T^k on V by pointwise multiplication

$$(\gamma z)(t) := \gamma(z(t)), \quad \gamma \in T^k, \quad z \in V, \quad t \in S^1.$$

Setting

$$N := \sum_{j=1}^n (m_j^+ - m_j^- + 1)$$

we will identify in the following V with \mathbb{C}^N using the map

$$z \mapsto (z_{1m_1^-}, \dots, z_{1m_1^+}, z_{2m_2^-}, \dots, z_{nm_n^+}).$$

Denote by μ_V the moment map associated by (3) to the action of T^k on V . The restriction of the action functional \mathcal{A}_0 to the finite dimensional space $V \times \mathfrak{t}^k$ is given by the formula

$$\mathcal{A}_0|_{V \times \mathfrak{t}^k}(z, \eta) = \int_0^1 \lambda(z)(\partial_t z) + \langle \eta, \mu_V(z) - \tau \rangle, \quad z \in V, \quad \eta \in \mathfrak{t}^k. \quad (16)$$

Denote by A_V the matrix corresponding to the moment map μ_V defined in (2). To each column vector of A_V corresponds a column vector of A , more precisely let $\rho: \{1, \dots, N\} \rightarrow \{1, \dots, n\}$ be defined by

$$\rho(k) := \min\{j : \sum_{i=1}^j (m_i^+ - m_i^- + 1) \geq k\}, \quad 1 \leq k \leq N$$

then

$$(A_V)_k = A_{\rho(k)}, \quad 1 \leq k \leq N.$$

Using this formula together with Remark 2.1 and Corollary 2.3 we obtain the following proposition.

Proposition 4.1 *Assume (H), i.e. the moment map μ is proper and τ is a regular value of μ . Then μ_V is also proper and τ is also a regular value of μ_V .*

Remark 4.2 *It follows from Proposition 4.1 that the spaces $\mu_V^{-1}(\tau)$ are compact manifolds. Their quotients under the torus action $\mu_V^{-1}(\tau)/T^k$ are therefore compact symplectic orbifolds. They are the toric map spaces introduced by Givental in [9] as an approximation to the moduli spaces of Floer homology for toric orbifolds, see also [8, 10, 16].*

We next examine the metrics g_0 and g_1 of \mathcal{L}_0 on the finite dimensional submanifold $V \times \mathfrak{t}^k$. The metric g_0 equals the metric induced by the standard scalar product on $V \times \mathfrak{t}^k \cong \mathbb{C}^N \times \mathfrak{t}^k$. It follows from (5) that g_1 on $V \times \mathfrak{t}^k$ is given by a product metric

$$g_1 = g_V \oplus g_{\mathfrak{t}^k}$$

where $g_{\mathfrak{t}^k}$ is induced by the standard scalar product on \mathfrak{t}^k and g_V is a Riemannian metric on V .

Proposition 4.3 *The Riemannian metric g_V on V is geodesically complete.*

Proof: We have to show that the length of each path $\gamma: [a, b) \rightarrow V \cong \mathbb{C}^N$ for $a, b \in \mathbb{R}$ such that $\gamma(s)$ tends to infinity as s goes to b has infinite length with respect to g_V . Define the path $\tilde{\gamma}: [0, \infty) \rightarrow \mathbb{C}^N$ as the reparametrisation of γ , which is parametrized by arclength with respect to the standard metric on \mathbb{C}^N . We estimate using Proposition 3.1

$$\begin{aligned} \text{length}(\gamma) &= \int_a^b \sqrt{g_V(\gamma(t))(\partial_t \gamma(t), \partial_t \gamma(t))} dt \\ &= \int_0^\infty \sqrt{g_V(\tilde{\gamma}(t))(\partial_t \tilde{\gamma}(t), \partial_t \tilde{\gamma}(t))} dt \\ &\geq \int_0^\infty \frac{1}{c(1 + \|\tilde{\gamma}(t)\|)} \|\partial_t \tilde{\gamma}(t)\| dt \\ &\geq \int_0^\infty \frac{1}{c(1 + \|\tilde{\gamma}(0)\| + t)} dt \\ &= \infty. \end{aligned}$$

This proves the proposition. \square

5 Conley index

Assume that M is a finite dimensional manifold and ϕ is a flow on M , i.e. a continuous map $\phi: M \times \mathbb{R} \rightarrow M, (x, t) \mapsto \phi^t(x)$, which satisfies $\phi^0 = \text{id}$ and $\phi^s \circ \phi^t = \phi^{s+t}$. For a subset $A \subset M$ we set

$$\text{Inv}_\phi A = \{x \in A : \phi^t(x) \in A, \forall t \in \mathbb{R}\}.$$

A subset $S \subset M$ is called an **isolated invariant set** if there exists an isolating neighbourhood A for S , i.e. a compact set A such that

$$S = \text{Inv}_\phi(A) \subset \text{int}(A).$$

In particular, S is compact and $S = \text{Inv}_\phi(S)$. The Conley index $I(\phi, S)$ attributes to every isolated invariant set S of the flow ϕ an isomorphism class of a pointed topological space, i.e. a topological space with a distinguished based point. We refer the reader to [4, 15] for an introduction into Conley index theory and to [12] for a short survey. We will use in this paper the following two properties of the Conley index.

Property 1 (Invariance) *Assume that A is an isolating neighbourhood for $S_r = \text{Inv}_{\phi_r} A$ for a continuous family of flows $\phi_r, r \in [0, 1]$, then $I(\phi_0, S_0) = I(\phi_1, S_1)$.*

To formulate the second property we have first to introduce some notation. Assume that $E \rightarrow M$ is a vector bundle. Choose a bundle metric g on E and denote by $B_g E$ and $S_g E$ the unit ball - respectively the unit sphere bundle - bundle over M . Then the Thom space of E is defined as the pointed topological space

$$T_g E := B_g E / S_g E := ((B_g E \setminus S_g E \cup [S_g E]), [S_g E])$$

obtained by collapsing $S_g E$ to a single point denoted by $[S_g E]$. A set $U \subset T_g E$ is open if either U is open in $B_g E$ and $U \cap S_g E = \emptyset$ or the set $(U \cap (B_g E \setminus S_g E)) \cup S_g E$ is open in $B_g E$. The pointed isomorphism class of $T_g E$ is independent of the choice of the bundle metric g and will be denoted by TE .

Now assume that ϕ is a gradient flow on the manifold M , i.e. there exists a function $f \in C^\infty(M, \mathbb{R})$ and a Riemannian metric g on M such that

$$\frac{d}{dt} \phi^t(x) = -\nabla_g f(\phi^t(x)), \quad x \in M.$$

Now assume that the subset $S \subset M$ is a critical submanifold of Morse-Bott type of the function f , i.e. S is a submanifold of M and for each $x \in S$

$$T_x S = \ker H_f(x)$$

where H_f denotes the Hessian of f . Under this assumption there is a natural splitting of the normal bundle NS of S in M

$$NS = N_f^+ S \oplus N_f^- S$$

where for $x \in S$, $(N_f^+ S)_x$ is spanned by the eigenvectors of the Hessian $H_f(x)$ to positive eigenvalues and $(N_f^- S)_x$ is spanned by the eigenvectors of $H_f(x)$ to negative eigenvalues.

We are now in the position to formulate the second property about the Conley index we will need.

Property 2 (Nontriviality) *Assume that ϕ is a gradient flow with respect to the function $f \in C^\infty(M, \mathbb{R})$ and S is a compact critical submanifold of Morse-Bott type of f . Then*

$$I(\phi, S) = T(N_f^- S).$$

In this section we are interested in some special class of isolated invariant sets. Assume that $X \in \Gamma(TM)$ is a vector field on M . For a flow line of X , i.e. a map $y \in C^\infty(\mathbb{R}, M)$ satisfying

$$\partial_s y(s) = X(y(s)), \quad s \in \mathbb{R}$$

we define its ω -limit sets as

$$\omega^+(y) = \bigcap_{t=0}^{\infty} \text{cl} \left(\bigcup_{s \geq t} y(s) \right), \quad \omega^-(y) = \bigcap_{t=0}^{\infty} \text{cl} \left(\bigcup_{s \geq t} y(-s) \right).$$

We denote by S_X the subset of M consisting of the traces of the flow lines of the vector field X both of whose ω -limit sets are nonempty, i.e.

$$S_X := \{y(\sigma) : \sigma \in \mathbb{R}, y \in C^\infty(\mathbb{R}, M), \partial_s y = X(y), \omega^\pm(y) \neq \emptyset\}.$$

Remark 5.1 *We point out that even for gradient vector fields the ω -limit sets may consist of more than one point, see [14]. However, if the critical set of a function consists of critical manifolds of Morse-Bott type then the assumption of nonempty ω -limit sets implies that the flow line converges at both ends. In our applications to the finite dimensional approximations of the action functional \mathcal{A}_0 we always have Morse-Bott situations, so that for these cases the set S_X may be defined as the traces of flow lines which converge at both ends.*

Assume that S_X is compact. If the manifold M is not compact then the vector field X may not generate a well-defined global flow. To overcome this problem choose compact subsets $A, A' \subset M$ such that $S_X \subset \text{int}(A) \subset A \subset A'$ and a smooth cutoff function β such that $\beta|_A = 1$ and $\beta|_{M \setminus A'} = 0$. Denote by $\phi_X = \phi_{X, \beta}$ the flow of the gradient of the vector field $\beta \cdot X$. Since $\beta \cdot X$ has compact support the flow ϕ_X is well defined. Moreover, the set S_X is an isolated invariant set with isolating neighbourhood A . We now set

$$I(X) = I(\phi_X, S_X).$$

It remains to check that $I(X)$ is indeed well-defined, i.e. independent of the choice of the cutoff function β . This follows readily from Property 1 of the Conley index. Indeed, assume that β_1 and β_2 are two cutoff functions satisfying the conditions above for compact sets A_1, A'_1 respectively A_2, A'_2 then for the family of flows $\{\phi_r\}, r \in [0, 1]$ of the vector field $(r\beta_1 + (1-r)\beta_2) \cdot X$ the set $A_1 \cap A_2$ is an isolating neighbourhood of S_X and hence the Conley indices for ϕ_0 and ϕ_1 agree by Property 1.

We are now in position to state the main theorem of this section.

Theorem 5.2 *Assume that (M, g) is a geodesically complete Riemannian manifold, $f \in C^\infty(M, \mathbb{R})$ and $h \in C^\infty(M, \mathbb{R}^k)$ for $k \in \mathbb{N}$ are smooth functions such that h is proper and 0 is a regular value of h . Define $F \in C^\infty(M \times \mathbb{R}^k, \mathbb{R})$ by*

$$F(x, \lambda) = f(x) + \langle h(x), \lambda \rangle, \quad x \in M, \lambda \in \mathbb{R}^k$$

and denote by ∇F its gradient with respect to the product metric $g \oplus g_{\mathbb{R}^k}$ on $M \times \mathbb{R}^k$ where $g_{\mathbb{R}^k}$ is the metric on \mathbb{R}^k obtained by the standard scalar product. Then $S_{-\nabla F}$ is compact and

$$I(-\nabla F) = T(N_M h^{-1}(0)) \quad (17)$$

where $N_M h^{-1}(0)$ denotes the normal bundle of $h^{-1}(0)$ in M . In particular, $I(-\nabla F)$ does not depend on f .

Definition 5.3 *A vector field $X \in \Gamma(TM)$ on a manifold M is called **tame** if there exists a compact set $K \subset M$, a constant $\epsilon > 0$, a geodesically complete Riemannian metric g on M , and a smooth function $F \in C^\infty(M, \mathbb{R})$ satisfying the following properties*

(i) *X does not vanish outside of K , i.e.*

$$X(x) = 0 \implies x \in K.$$

(ii) *F is a Lyapunov function, i.e.*

$$dF(x)X(x) \leq 0, \quad x \in M.$$

(iii) *X satisfies the following Palais-Smale condition*

$$dF(x)X(x) \leq -\epsilon \sqrt{g(X(x), X(x))}, \quad x \in M \setminus K.$$

A smooth family of vector fields $X_r \in \Gamma(TM)$ for $r \in [0, 1]$ is called tame if there exist a compact set $K \subset M$, a constant $\epsilon > 0$, and a geodesically complete Riemannian metric g independent of r , and a family of smooth functions $F_r \in C^\infty(M, \mathbb{R})$ which are uniformly bounded on K satisfying conditions (i), (ii), and (iii) above for every $r \in [0, 1]$.

Proposition 5.4 *Assume that $X \in \Gamma(TM)$ is a tame vector field. Then S_X is compact. Moreover, if $X_r \in \Gamma(TM)$ for $r \in [0, 1]$ is a tame family of vector fields then $I(X_r)$ does not depend on $r \in [0, 1]$.*

Proof: Assume that $y \in C^\infty(\mathbb{R}, M)$ is a flow line of X , i.e.

$$\partial_s y = X(y),$$

such that $\omega^\pm(y) \neq \emptyset$. We show that there exists a compact set $K' \subset M$ independent of y such that the trace of y is entirely contained in K' , i.e.

$$y(\sigma) \in K', \quad \sigma \in \mathbb{R}.$$

Choose $y^\pm \in \omega^\pm(y)$. It follows from the Lyapunov property (ii) in Definition 5.3 that $dF(y^\pm)X(y^\pm) = 0$. Now properties (i) and (iii) imply that $y^\pm \in K$. Since K is compact there exists a constant $m > 0$ independent of y such that

$$|F(y^\pm)| \leq m.$$

Using again (ii) we conclude that

$$|F(y(\sigma))| \leq m, \quad \sigma \in \mathbb{R}.$$

For $s \in \mathbb{R}$ define

$$\tau(s) := \inf_{s' \geq s} \{y(s') \in K\}.$$

Here we use the convention that the infimum of the empty set equals infinity. We now estimate the distance from $y(s)$ to K by

$$\begin{aligned} d(y(s), K) &\leq \int_s^{\tau(s)} \|\dot{y}(\sigma)\| d\sigma \\ &= \int_s^{\tau(s)} \|X(y(\sigma))\| d\sigma \\ &\leq -\frac{1}{\epsilon} \int_s^{\tau(s)} dF(y(\sigma))X(y(\sigma)) d\sigma \\ &\leq -\frac{1}{\epsilon} \int_{-\infty}^{\infty} dF(y(\sigma))X(y(\sigma)) d\sigma \\ &= -\frac{1}{\epsilon} \int_{-\infty}^{\infty} \frac{d}{d\sigma} F(y(\sigma)) d\sigma \\ &= \lim_{s \rightarrow \infty} \frac{1}{\epsilon} \left(F(y(-s)) - F(y(s)) \right) \\ &\leq \frac{2m}{\epsilon}. \end{aligned}$$

We now define

$$K' = \{x \in M : d(x, K) \leq 2m/\epsilon\}.$$

Since the metric g is geodesically complete the set K' is compact. Hence the set S_X is compact as a closed subset of the compact set K' .

If $X_r \in \Gamma(TM)$ for $r \in [0, 1]$ is a tame family of vector fields, we define K' as above, by choosing as m the uniform bound of the family of functions F_r on K . The above estimate then shows that S_{X_r} is contained in K for every $r \in [0, 1]$. Now choose compact sets A, A' such that $K \subset \text{int}(A) \subset A \subset A'$ and a smooth cutoff function β such that $\beta|_A = 1$ and $\beta_{M \setminus A'} = 0$. Define ϕ_r to be the flow of the compactly supported vector field $\beta \cdot X_r$ for $r \in [0, 1]$. Then A is an isolating neighbourhood for $S_{X_r} = \text{Inv}_{\phi_r} A$ and the invariance of the Conley indices $I(X_r)$ follows from Property 1. This proves the Proposition. \square

Lemma 5.5 *Assume that (M, g) is a geodesically complete Riemannian manifold and let $f \in C^\infty(M, \mathbb{R})$ and $h \in C^\infty(M, \mathbb{R}^k)$ for $k \in \mathbb{N}$ satisfy the assumptions of Theorem 5.2. For $r \in [0, 1]$ define the functions $F_r \in C^\infty(M \times \mathbb{R}^k)$ by*

$$F_r(x, \lambda) := rf(x) + \langle \lambda, h(x) \rangle, \quad x \in M, \lambda \in \mathbb{R}^k.$$

Denote by ∇F_r the gradient of F_r with respect to the product metric $g \oplus g_{\mathbb{R}^k}$. Then the family of vector fields $X_r = -\nabla F_r$ is tame.

Proof: Since h is proper and 0 is a regular value of h there exists $\epsilon > 0$ such that every $\lambda \in B_\epsilon = \{\mu \in \mathbb{R}^k : \|\mu\| \leq \epsilon\}$ is a regular value of h . Consider the continuous function

$$\rho: h^{-1}(B_\epsilon) \rightarrow \mathbb{R}, \quad x \mapsto \min \left\{ \left\| \sum_{i=1}^k \lambda_i \nabla h_i(x) \right\| : \lambda \in \mathbb{R}^k, \|\lambda\| = 1 \right\}$$

where h_i for $1 \leq i \leq k$ denotes the i -th component of the function $h \in C^\infty(M, \mathbb{R}^k)$. Since $dh(x)$ is surjective for every $x \in h^{-1}(B_\epsilon)$ it follows that $\rho(x) > 0$ and hence

$$\delta := \min\{\rho(x) : x \in h^{-1}(B_\epsilon)\} > 0.$$

Set

$$c := \max\{\|\nabla f(x)\| : x \in h^{-1}(B_\epsilon)\}$$

and define

$$K := \left\{ (x, \lambda) \in M \times \mathbb{R}^k : x \in h^{-1}(B_\epsilon), \|\lambda\| \leq \frac{c + \epsilon}{\delta} \right\}.$$

The set K is compact and we claim

$$\|\nabla F_r(x, \lambda)\| > \epsilon, \quad r \in [0, 1], (x, \lambda) \in M \times \mathbb{R}^k \setminus K. \quad (18)$$

In order to prove (18) we first compute the gradient of F_r

$$\nabla F_r(x, \lambda) = \left(r \cdot \nabla f(x) + \sum_{i=1}^k \lambda_i \cdot \nabla h_i(x) \right).$$

We first consider the case where $x \notin h^{-1}(B_\epsilon)$. Then

$$\|\nabla F_r(x, \lambda)\| \geq \|h(x)\| > \epsilon$$

and (18) holds. Now consider the case where $x \in h^{-1}(B_\epsilon)$ but $\|\lambda\| > (c + \epsilon)/\delta$. We estimate for every $r \in [0, 1]$

$$\begin{aligned} \|\nabla F_r(x, \lambda)\| &\geq \|r \cdot \nabla f(x) + \sum_{i=1}^k \lambda_i \cdot \nabla h_i(x)\| \\ &> \left\| \sum_{i=1}^k \lambda_i \cdot \nabla h_i(x) \right\| - \|r \cdot \nabla f(x)\| \\ &\geq \delta \cdot \left(\frac{c + \epsilon}{\delta} \right) - c \\ &= \epsilon \end{aligned}$$

which proves (18). The lemma now follows for the obvious choices of the ingredients in the definition of tame family, namely choose K as the compact set in $M \times \mathbb{R}$, choose ϵ as the positive constant, choose $g \oplus g_0$ as the geodesically complete Riemannian metric, and choose F_r itself as the family of functions. \square

Proof of Theorem 5.2: We consider the homotopy of functions

$$F_r(x, \lambda) := rf(x) + \langle \lambda, h(x) \rangle, \quad x \in M, \quad \lambda \in \mathbb{R}^k$$

for $r \in [0, 1]$. Note that $F_1 = F$. It follows from Proposition 5.4 and Lemma 5.5 that

$$I(-\nabla F) = I(-\nabla F_0). \quad (19)$$

It remains to compute $I(-\nabla F_0)$. Using the formula for the gradient of $F_0(x, \lambda) = \langle \lambda, h(x) \rangle$

$$\nabla F_0(x, \lambda) = \begin{pmatrix} \sum_{i=1}^k \lambda_i \cdot \nabla h_i(x) \\ h(x) \end{pmatrix}$$

and the fact that 0 is a regular value of h we conclude that the critical set C of F_0 is a manifold and consists of

$$C = \{(x, 0) : h(x) = 0\}.$$

In particular, F_0 is constant equal 0 on C and hence all flow lines whose limits exist on both sides are constant paths in C . The Hessian H_{F_0} at a point $(x, 0) \in C$ is given by

$$H_{F_0}(x, 0) = \begin{pmatrix} 0 & dh(x)^* \\ dh(x) & 0 \end{pmatrix} \quad (20)$$

where the adjoint $dh(x)^*$ is taken with respect to the metric g_x on $T_x M$ and the standard scalar product on $T_{h(x)} \mathbb{R}^k \cong \mathbb{R}^k$. Using the assumption that 0 is a regular value of h we conclude that

$$\text{rk}(H_{F_0}(x, 0)) = 2k = \text{codim}_{M \times \mathbb{R}^k} C.$$

This shows that C is a critical submanifold of Morse-Bott type. The normal bundle NC of C in $M \times \mathbb{R}^k$ splits into the sum

$$NC = N_{F_0}^+ C \oplus N_{F_0}^- C$$

given by the positive, respectively negative, eigenvalues of the Hessian H_{F_0} and we conclude from property 2 of the Conley index that

$$I(-\nabla F_0) = T(N_{F_0}^- C). \quad (21)$$

In view of Lemma 5.6 below we conclude from (19) and (21) that (17) holds, which proves the theorem. \square

Lemma 5.6 *The normal bundle $N_M h^{-1}(0)$ of $h^{-1}(0)$ in M and the bundle $N_{F_0}^- C$ are isomorphic vector bundles.*

There is a natural diffeomorphism from C to $h^{-1}(0)$ given by the map $(x, 0) \mapsto x$. Using (20) we conclude that $N_{F_0}^- C$ at $(x, 0)$ is spanned by the vectors

$$w_v := \begin{pmatrix} v \\ -\frac{1}{\lambda} dh(x)v \end{pmatrix}$$

where v is an eigenvector of $dh(x)^* dh(x)$ to the eigenvalue λ^2 , i.e.

$$dh(x)^* dh(x)v = \lambda^2 v.$$

We can now identify the vector bundle $N_{F_0}^- C$ over $(x, 0)$ with the normal bundle $N_M h^{-1}(0)$ over x by the linear extension of the map

$$w_v \mapsto v.$$

This finishes the proof of the lemma. \square

For a finite dimensional approximation V of the loop space $C^\infty(S^1, \mathbb{C}^n)$ we define $I_V^{g_i}$ for $i \in \{0, 1\}$ as the Conley index for the gradient of $\mathcal{A}_0|_{V \times \mathfrak{t}^k}$ with respect to the metric g_i . The proof of Theorem A is now straightforward.

Proof of Theorem A: It follows from formula (16) that the action functional $\mathcal{A}_0|_{V \times \mathfrak{t}^k}$ is the sum of the restriction of Floer's action functional to V and a Lagrange multiplier to the condition $\mu_V^{-1}(\tau)$. It follows from Proposition 4.1 that the map $\mu_V - \tau$ is proper and 0 is a regular value of it. The metric g_0 is obviously geodesically complete on V and for the metric g_1 this follows from Proposition 4.3. Theorem A follows now from Theorem 5.2. \square

6 Cobordism

In this section we prove Theorem B and Theorem C. Theorem B follows from the following Theorem.

Theorem 6.1 *Let $(z, \eta) \in C^\infty(\mathbb{R} \times S^1, \mathbb{C}^n) \times C^\infty(\mathbb{R}, \mathfrak{t}^k)$ be a gradient flow line of \mathcal{A}_0 with respect to the metric g_0 on \mathcal{L}_0 , i.e.*

$$\partial_s(z, \eta)(s, \cdot) = -\nabla_0 \mathcal{A}_0((z, \eta)(s, \cdot)), \quad s \in \mathbb{R}.$$

Assume the flow line converges on both ends, i.e.

$$\lim_{s \rightarrow \pm\infty} (z, \eta)(s) = (z^\pm, \eta^\pm) \in \mathcal{L}_0,$$

where the limits are taken with respect to the C^∞ -topology, exist. Denote for $1 \leq j \leq n$

$$m_j^- := \min_{n \in \mathbb{Z}} \{2\pi n \geq \sum_{r=1}^k A_{jr} \eta_r^-\}, \quad m_j^+ := \max_{n \in \mathbb{Z}} \{2\pi n \leq \sum_{r=1}^k A_{jr} \eta_r^+\},$$

where A is as usual the $n \times k$ -matrix which defines the action of the torus on \mathbb{C}^n . Then the trace of the flow line (z, η) is contained in the finite dimensional subspace $V = V_{\{m_j^-, m_j^+\}_{1 \leq j \leq n}}$ of \mathcal{L}_0 , i.e.

$$(z, \eta)(s) \in V, \quad \forall s \in \mathbb{R}.$$

Proof: The proof is based on Fourierapproximation. Let

$$z_j(s, t) = \sum_{m=-\infty}^{\infty} z_{jm}(s) e^{2\pi i m t}, \quad 1 \leq j \leq n.$$

The formula (11) shows that for $j \in \{1, \dots, n\}$ and $m \in \mathbb{Z}$

$$\partial_s z_{jm}(s) + \left(\sum_{r=1}^k A_{jr} \eta_r(s) - 2\pi m \right) z_{jm}(s) = 0.$$

Using $\lim_{s \rightarrow \pm\infty} \partial_s z_{jm}(s) = 0$ we conclude that z_{jm} vanishes identically if m is not contained in $\{m_j^-, \dots, m_j^+\}$. \square

Our next aim is to prove Theorem C. To motivate our construction of the homotopy of metrics g_r on \mathcal{L}_0 for $r \in [0, 1]$ it is useful to consider the family of action functionals

$$\mathcal{A}^r : \mathcal{L} = C^\infty(S^1, \mathbb{C}^n \times \mathfrak{t}^k) \rightarrow \mathbb{R}, \quad r \in [0, 1]$$

given by

$$\mathcal{A}^r(z, \eta) := \int_0^1 \lambda(z)(\partial_t z) + \int_0^1 \langle (1-r)\mu(z(t)) + r\bar{\mu}(z) - \tau, \eta(t) \rangle dt.$$

Note that

$$\mathcal{A}^0 = \mathcal{A}, \quad \mathcal{A}^r|_{\mathcal{L}_0} = \mathcal{A}_0, \quad r \in [0, 1].$$

The action functionals \mathcal{A}^r for $r \in [0, 1]$ are invariant under the usual action of the finite dimensional group H and for $r \in (0, 1]$ they are also invariant under the following deformed action of the gauge group \mathcal{H}_0 on \mathcal{L}

$$h_{*r}(z, \eta) = (\rho(h)z, \eta - \frac{1}{r}h^{-1}\partial_t h), \quad h \in \mathcal{H}_0, \quad (z, \eta) \in \mathcal{L}.$$

Note that for each $r \in (0, 1]$ the deformed action of \mathcal{H}_0 is free on \mathcal{L} and for each point of \mathcal{L}_0 there is exactly one gauge orbit which goes through this point. We now define g_r for $r \in (0, 1]$ to be the quotient metric on $\mathcal{L}_0 \cong_r \mathcal{L}/_r \mathcal{H}_0$ of the L^2 -metric g on \mathcal{L} . Here the r -parameter takes account of the r -dependence of the action of \mathcal{H}_0 on \mathcal{L} . We denote by $\nabla_r \mathcal{A}_0$ the gradient of \mathcal{A}_0 with respect to the metric g_r . To compute it, note that the gradient of \mathcal{A}^r with respect to the L^2 -metric g on \mathcal{L} for $r \in [0, 1]$ is given by

$$\nabla \mathcal{A}^r(z, \eta) = \begin{pmatrix} i\partial_t z + (1-r)iX_{\bar{\eta}}(z) + riX_{\eta}(z) \\ (1-r)\bar{\mu}(z) + r\mu(z) \end{pmatrix}.$$

As in the proof of formula (12) one shows that for $(z, \eta) \in \mathcal{L}_0$

$$\nabla_r \mathcal{A}_0(z, \eta) = (i\partial_t z + iL_z \eta + L_z \xi, \bar{\mu}(z) - \tau)$$

where $\xi \in \text{Lie}(\mathcal{H}_0)$ is determined by

$$\partial_t \xi(t) = r^2(\mu(z(t)) - \bar{\mu}(z)), \quad \int_0^1 \xi dt = 0.$$

In particular,

$$\nabla_r \mathcal{A}_0(z, \eta) = r^2 \nabla_0 \mathcal{A}_0(z, \eta) + (1-r^2) \nabla_1 \mathcal{A}_0(z, \eta), \quad r \in [0, 1].$$

The energy of a flow line is defined to be

$$E(z, \eta) = \int_{-\infty}^{\infty} \int_0^1 (|\partial_t z + L_z \eta|^2 + |\bar{\mu}(z) - \tau|^2) dt ds.$$

In order to prove Theorem C we first show the following lemma which provides us with a uniform L^∞ -bound for z_ν .

Lemma 6.2 *There exists a compact set $K \subset \mathbb{C}^n$ such that for each flow line (z, η) of \mathcal{A}_0^r for $r \in (0, 1]$ whose energy is bounded, the trace of z is contained in K , i.e.*

$$z(s, t) \in K, \quad s \in \mathbb{R}, \quad t \in S^1.$$

Proof: We first examine the Laplacian of the function $|z|^2/2$. Observe that by (3) and Remark 2.1 for $z, z' \in \mathbb{C}^n$

$$\langle \mu(z), \mu(z') \rangle \geq 0.$$

Using this inequality and a computation similar to the one in the proof of [2, Proposition 3.5] we estimate

$$\begin{aligned}
\Delta|z|^2/2 &= |\partial_s z + L_z \xi|^2 + |\partial_t z + L_z \eta|^2 \\
&\quad + 2\langle \mu(z), (r^2 - 1)\bar{\mu}(z) + r^2\mu(z) - \tau \rangle \\
&\geq 2r^2\langle \mu(z), \mu(z) - \tau \rangle \\
&\geq 2r^2|\mu(z)|(|\mu(z)| - |\tau|).
\end{aligned} \tag{22}$$

We set

$$R := \max_{|\mu(z)| \leq \tau} \{|z|\}, \quad K := \{z \in \mathbb{C}^n : |z| \leq 2R\}.$$

We assume by contradiction that there exists $(s_0, t_0) \in \mathbb{R} \times S^1$ such that

$$z(s_0, t_0) \in \mathbb{C}^n \setminus K.$$

By the convexity property derived in (22) it follows that either for all $s \geq s_0$ or for all $s \leq s_0$ there exists t_s with

$$|z(s, t_s)| \geq |z(s_0, t_0)|.$$

We consider only the case where the above property holds for all $s \geq s_0$, the treatment of the other case is completely analogous. Using the assumption that the energy is bounded we conclude that there exist for each $i \in \mathbb{N}$ real numbers $\sigma_i < \sigma^i$ with the property that

$$\sigma_1 \geq s_0, \quad \sigma^i \leq \sigma_{i+1} \quad \forall i \in \mathbb{N}, \quad \sum_{i=1}^{\infty} (\sigma^i - \sigma_i) = \infty$$

such that for each $s \in [\sigma_i, \sigma^i]$ for $i \in \mathbb{N}$ it exists $t'_s \in S^1$ such that

$$z(s, t'_s) \in K.$$

Abbreviating

$$\delta := |z(s_0, t_0)| - R > 0$$

we estimate

$$\begin{aligned}
\infty &= \sum_{i=1}^{\infty} \delta^2(\sigma^i - \sigma_i) \\
&\leq \sum_{i=1}^{\infty} \int_{\sigma_i}^{\sigma^i} \left(\int_{t'_s}^{t_s} \|\partial_t z + L_z \eta\| dt \right)^2 ds \\
&\leq \sum_{i=1}^{\infty} \int_{\sigma_i}^{\sigma^i} \left(\int_0^1 \|\partial_t z + L_z \eta\| dt \right)^2 ds \\
&\leq \sum_{i=1}^{\infty} \int_{\sigma_i}^{\sigma^i} \int_0^1 \|\partial_t z + L_z \eta\|^2 dt ds \\
&\leq \int_{-\infty}^{\infty} \int_0^1 \|\partial_t z + L_z \eta\|^2 dt ds \\
&\leq E(z, \eta) \\
&< \infty.
\end{aligned}$$

This contradiction proves the lemma. \square

Proof of Theorem C: By Theorem 6.1 we may assume without loss of generality that $r_\nu > 0$ for every $\nu \in \mathbb{N}$. Choose $h_\nu \in H$ such that

$$0 \leq (h_\nu)_*(\eta_\nu)_i(0) < 2\pi, \quad 1 \leq i \leq k.$$

We will replace in the following (z_ν, η_ν) by $(h_\nu)_*(z_\nu, \eta_\nu)$. By Lemma 6.2 the maps z_ν are uniformly bounded in the L^∞ -topology. The proof of Theorem C reduces now to elliptic bootstrapping for the equations

$$\begin{aligned}
\partial_s z_\nu + i\partial_t z_\nu + L_{z_\nu} \xi_\nu + iL_{z_\nu} \eta_\nu &= 0 \\
\partial_s \eta_\nu + \bar{\mu}(z_\nu) &= \tau \\
\partial_t \xi_\nu &= r_\nu^2 (\mu(z_\nu) - \bar{\mu}(z_\nu)) \\
\int_0^1 \xi_\nu dt &= 0.
\end{aligned} \tag{23}$$

Fix $N \in \mathbb{N}$. Using the fact that there exists a constant $c_0 = c_0(N)$ such that

$$\|\partial_s z_\nu + i\partial_t z_\nu\|_{L^2([-N, N] \times S^1)} \leq c_0 \|z_\nu\|_{W^{1,2}([-N-1, N+1] \times S^1)}$$

we deduce from the uniform L^∞ -bound on z_ν and the uniform bound on $\eta_\nu(0)$ from (23) that there exists $c_1 > 0$ such that

$$\|z_\nu\|_{W^{1,2}([-N, N] \times S^1)} + \|\eta_\nu\|_{L^2([-N, N])} + \|\xi_\nu\|_{L^2([-N, N] \times S^1)} \leq c_1. \tag{24}$$

In order to get control over the higher derivatives it is useful to use the original \mathcal{H}_0 -action on \mathcal{L} to put (23) into Coulomb gauge on the cylinder. Let $\zeta_\nu \in$

$C^\infty([-N, N] \times S^1, \mathfrak{t}^k)$ be a solution of the following Neumann problem on the finite cylinder $[-N, N] \times S^1$ with mean value zero

$$\begin{aligned}\Delta \zeta_\nu(s, t) &= r_\nu^2 d(\mu(z_\nu) - \bar{\mu}(z_\nu)) \partial_s z_\nu(s, t) = \partial_s \partial_t \xi_\nu, \quad s \in [-N, N], \quad t \in S^1, \\ \partial_s \zeta_\nu(\pm N, t) &= 0, \quad t \in S^1, \\ \int_0^1 \zeta_\nu(s, t) dt &= 0, \quad s \in [-N, N].\end{aligned}$$

Define $g_\nu \in C^\infty([-N, N] \times S^1, T^k)$ as the map from the interval $[-N, N]$ to the gauge group \mathcal{H}_0 which is defined by the property

$$\partial_t(g_\nu^{-1} \partial_t g_\nu) = \zeta_\nu.$$

Define $(\tilde{z}_\nu, \tilde{\eta}_\nu, \tilde{\xi}_\nu) \in C^\infty([-N, N] \times S^1, \mathbb{C} \times \mathfrak{t}^k \times \mathfrak{t}^k)$ by

$$(\tilde{z}_\nu, \tilde{\eta}_\nu, \tilde{\xi}_\nu) := (g_\nu)_*(z_\nu, \eta_\nu, \xi_\nu) := (\rho(g_\nu)z_\nu, \eta_\nu - (g_\nu)^{-1} \partial_t g_\nu, \xi_\nu - (g_\nu)^{-1} \partial_s g_\nu).$$

Then $(\tilde{z}_\nu, \tilde{\eta}_\nu, \tilde{\xi}_\nu)$ is a solution of the problem

$$\begin{aligned}\partial_s \tilde{z}_\nu + i \partial_t \tilde{z}_\nu + L_{\tilde{z}_\nu} \tilde{\xi}_\nu + i L_{\tilde{z}_\nu} \tilde{\eta}_\nu &= 0 \\ \partial_s \tilde{\eta}_\nu - \partial_t \tilde{\xi}_\nu + (1 - r_\nu^2) \bar{\mu}(z_\nu) + r_\nu^2 \mu(z_\nu) &= \tau \\ \partial_s \tilde{\xi}_\nu + \partial_t \tilde{\eta}_\nu &= 0 \\ \int_0^1 \tilde{\xi}_\nu dt &= 0.\end{aligned}\tag{25}$$

Combining (24) with Lemma 6.3 below we see that there exists a constant $c_2 > 0$ such that

$$\|\tilde{z}_\nu\|_{W^{1,2}([-N, N] \times S^1)} + \|\tilde{\eta}_\nu\|_{L^2([-N, N] \times S^1)} + \|\tilde{\xi}_\nu\|_{L^2([-N, N] \times S^1)} \leq c_2. \tag{26}$$

By elliptic regularity for the Cauchy-Riemann operator it follows from (25) and (26) that $(\tilde{z}_\nu, \tilde{\eta}_\nu, \tilde{\xi}_\nu)$ is uniformly bounded on $[-N+1, N-1] \times S^1$ in the $W^{k,2}$ -norm for every $k \in \mathbb{N}$. By the Sobolev embedding theorem and the theorem of Rellich it follows that there exists a subsequence ν_j such that

$$(\tilde{z}_{\nu_j}, \tilde{\eta}_{\nu_j}, \tilde{\xi}_{\nu_j})|_{[-N+1, N-1] \times S^1} \xrightarrow{C^\infty, j \rightarrow \infty} (\tilde{z}, \tilde{\eta}, \tilde{\xi})$$

where $(\tilde{z}, \tilde{\eta}, \tilde{\xi})$ is a solution of (25) on $[-N+1, N-1] \times S^1$ for some $r \in [0, 1]$. Define $g \in C^\infty([-N+1, N-1] \times S^1, T^k)$ to be the map from $[-N+1, N-1]$ to \mathcal{H}_0 defined by the property that $g_* \tilde{\eta}$ is independent of the t -variable. Then

$$(z, \eta, \xi) = g_*(\tilde{z}, \tilde{\eta}, \tilde{\xi})$$

is a solution of (23). Moreover,

$$(g \circ g_{\nu_j})_*(z_{\nu_j}, \eta_{\nu_j}, \xi_{\nu_j})|_{[-N+1, N-1] \times S^1} \xrightarrow{C^\infty, j \rightarrow \infty} (z, \eta, \xi). \tag{27}$$

Since η_{ν_j} for every $j \in \mathbb{N}$ and η are independent of the t -variable it follows that

$$\partial_t((g \circ g_{\nu_j})^{-1} \partial_t(g \circ g_{\nu_j})) \xrightarrow{C^\infty}_{j \rightarrow \infty} 0.$$

It follows that

$$g \circ g_{\nu_j} \xrightarrow{C^\infty}_{j \rightarrow \infty} \text{id}. \quad (28)$$

Combining (27) and (28) we conclude that

$$(z_{\nu_j}, \eta_{\nu_j}, \xi_{\nu_j})|_{[-N+1, N-1] \times S^1} \xrightarrow{C^\infty}_{j \rightarrow \infty} (z, \eta, \xi).$$

Since $N \in \mathbb{N}$ was arbitrary the theorem follows. \square

Lemma 6.3 *There exists a constant $c > 0$ with the following property. Assume that $g_\pm \in C^\infty(S^1, \mathbb{R})$ and $h \in C^\infty([-N, N] \times S^1, \mathbb{R})$ for $N \in \mathbb{N}$ satisfy*

$$\int_0^1 g_\pm(t) dt = 0, \quad \int_0^1 h(s, t) dt = 0, \quad s \in [-N, N].$$

Suppose that $f \in C^\infty([-N, N] \times S^1, \mathbb{R})$ is a solution of the Neumann problem on the finite cylinder $[-N, N] \times S^1$ with mean value zero

$$\Delta f(s, t) = h(s, t), \quad s \in [-N, N], \quad t \in S^1, \quad \pm \partial_s f(\pm N, t) = g_\pm(t), \quad t \in S^1,$$

$$\int_0^1 f(s, t) dt = 0, \quad s \in [-N, N],$$

then

$$\|f\|_{W^{2,2}([-N, N] \times S^1)} \leq c(\|h\|_{L^2([-N, N] \times S^1)} + \|g_+\|_{W^{1,2}(S^1)} + \|g_-\|_{W^{1,2}(S^1)}).$$

Proof: The proof is an exercise in partial integration. We estimate

$$\begin{aligned} & \int_{-N}^N \int_0^1 ((\partial_s^2 f)^2 + (\partial_t^2 f)^2 + 2(\partial_s \partial_t f)^2) dt ds \\ &= \int_{-N}^N \int_0^1 ((\partial_s^2 f)^2 + (\partial_t^2 f)^2 - 2(\partial_s^2 \partial_t f)(\partial_t f)) dt ds \\ & \quad + \int_0^1 \partial_t g_+(t) \partial_t f(N, t) dt + \int_0^1 \partial_t g_-(t) \partial_t f(-N, t) dt \\ &\leq \int_{-N}^N \int_0^1 ((\partial_s^2 f)^2 + (\partial_t^2 f)^2 + 2(\partial_s^2 f)(\partial_t^2 f)) dt ds \\ & \quad + \|\partial_t g_+\|_2 \|\partial_t f(N, \cdot)\|_2 + \|\partial_t g_-\|_2 \|\partial_t f(-N, \cdot)\|_2 \\ &\leq \|h\|_2^2 + 16\|g_+\|_{1,2}^2 + 16\|g_-\|_{1,2}^2 + \frac{1}{16} \|\partial_t f(N, \cdot)\|_2^2 + \frac{1}{16} \|\partial_t f(-N, \cdot)\|_2^2. \end{aligned} \quad (29)$$

Using the assumption that the mean value of $f(s, \cdot)$ vanishes for each $s \in [-N, N]$ we obtain the estimates

$$\|f\|_2 \leq \|\partial_t f\|_2 \leq \|\partial_t^2 f\|_2, \quad \|\partial_s f\|_2 \leq \|\partial_s \partial_t f\|_2. \quad (30)$$

Choose a smooth function $\beta: [-N, N] \rightarrow \mathbb{R}$ which satisfies

$$\begin{aligned}\beta(s) &= 0, \quad s \leq N-1, \quad \beta(N) = 1, \quad 0 \leq \beta(s) \leq 1, \quad s \in [-N, N], \\ \beta'(s) &\leq 2, \quad s \in [-N, N].\end{aligned}$$

We estimate

$$\begin{aligned}\|\partial_t f(N, \cdot)\|_2^2 &= \int_{N-1}^N \partial_s \left(\beta(s) \int_0^1 (\partial_t f)^2(s, t) dt \right) ds \\ &= \int_{N-1}^N \int_0^1 \beta'(s) (\partial_t f)^2(s, t) ds dt \\ &\quad + \int_{N-1}^N \int_0^1 2\beta(s) \partial_s \partial_t f(s, t) \partial_t f(s, t) ds dt \\ &\leq 4\|\partial_t f\|_2^2 + 2\|\partial_s \partial_t f\|_2^2 \\ &\leq 4\|\partial_t^2 f\|_2^2 + 2\|\partial_s \partial_t f\|_2^2.\end{aligned}\tag{31}$$

For the last inequality we have used (30). Similarly one shows

$$\|\partial_t f(-N, \cdot)\|_2^2 \leq 4\|\partial_t^2 f\|_2^2 + 2\|\partial_s \partial_t f\|_2^2.\tag{32}$$

Combining (29) with (31) and (32) one obtains

$$\|\partial_s^2 f\|_2^2 + \|\partial_t^2 f\|_2^2 + \|\partial_s \partial_t f\|_2^2 \leq 2\|h\|_2^2 + 32\|g_+\|_{1,2}^2 + 32\|g_-\|_{1,2}^2.\tag{33}$$

The lemma follows now by combining (33) with (30). \square

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